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The symmetric group: branching rules, products and plethysms for spin representations

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Abstract. It is shown that the symmetric group S_n may be usefully embedded in the orthogonal group O_n, and that this embedding leads directly to an n-independent 'reduced' notation for both the spin and ordinary representations of S_n. Making use of this embedding, together with the properties of Q-functions (or Hall–Littlewood functions), branching rules for $O_n \downarrow S_n$ are developed and the general rule for the decomposition of spin representations under $S_n \downarrow S_{n-1}$ is obtained. Simple methods are given for calculating all possible Kronecker products involving the spin and ordinary representations of S_n and the resolution of Kronecker squares into their symmetric and antisymmetric parts. The spin representations of S_n are systematically classified as to their orthogonal, symplectic or complex characters. The emphasis throughout is on obtaining results that obviate the need for explicit character tables and presenting results in an n-independent manner as much as possible.

1. Introduction

The symmetric group S_n has long been of interest to physicists and chemists who have sought to exploit the permutational symmetry associated with many-fermion and many-boson systems. These applications have usually made use of the well developed theory of the ordinary representations of S_n (Murnaghan 1938, Littlewood 1950, Robinson 1961). The problem of resolving the Kronecker products of the ordinary representations of S_n has received considerable attention, and techniques have been developed that obviate the need to use explicit character tables (cf Murnaghan 1937, 1938, Littlewood 1958a, b, Butler and King 1973). Furthermore, many of the results have been given in an n-independent form using a 'reduced' notation for labelling the irreducible representations (irreps) of S_n.

The spin (or projective) irreps of S_n have received far less attention, although physicists are familiar with the spin representations of crystallographic point groups. As long ago as 1911 Issia Schur, having previously investigated the representations of any finite group by linear fractional substitutions (Schur 1904, 1907), directed his attention to the study of the linear fractional substitution representation group $\mathfrak{T}_n$ of S_n (Schur 1911). This group was later shown to be isomorphic to the $2(n!)$-order group Γ_n, called the *spin group* of S_n (Morris 1962a, b). The ordinary characters of S_n are also characters of Γ_n and the remaining characters of Γ_n are known as the *spin characters* of S_n.

Methods of constructing spin character tables of S_n are of recent origin (cf Morris 1962a, Read 1977). Remarkably little is known about the resolution of Kronecker

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products involving the spin representations apart from the explicit use of character tables. This contrasts strongly with the corresponding results known for the ordinary representations of S_n. While the branching rule for the reduction of the ordinary irreps under $S_n \downarrow S_{n-1}$ is well known (Boerner 1970), the corresponding rule for the spin representations would still appear to be incomplete (cf Wales 1979).

In this paper we shall first review some relevant, though possibly unfamiliar, aspects of the ordinary representations of S_n and then consider some of the properties of the spin representations of S_n. We then present some simple results relevant to associated and self-associated representations and establish an $O_n \supset S_n$ embedding. The formation of branching rules for $O_n \downarrow S_n$ is then considered, leading to a 'reduced' notation for the spin representations of S_n, making possible many n-independent results. The first application is to discuss the n-independence of the dimensions of the spin representations of S_n. The properties of the Q-functions are then exploited to give the $S_n \downarrow S_{n-1}$ branching theorem. An inner multiplication of Q-functions with S-functions leads to a simple procedure for resolving the Kronecker product of the basic spin representation of S_n with any ordinary representation of S_n into spin representations of S_n. These results, together with consideration of the difference characters of S_n, give a general procedure for resolving arbitrary Kronecker products without the explicit use of character tables. We are then able to use the method of plethysm to resolve Kronecker squares of the spin representations into their symmetric and antisymmetric parts, and eventually to classify the spin irreps as to their orthogonal, symplectic or complex characters. The application of these results to the problem of calculating the nj and $3jm$ symbols associated with the spin and ordinary irreps of S_n is briefly considered.

The results given in this paper remove many of the past difficulties associated with the spin representations of S_n, and are presented in the hope that they will stimulate applications of these important groups to physical problems.

2. Ordinary representations of S_n

The ordinary irreps of S_n may be uniquely labelled by the ordered integer partitions $(\lambda) \equiv (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of the integer n, i.e.

$$\lambda_1+\lambda_2+\ldots+\lambda_k=n \qquad \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k \geqslant 0.$$

These representations of S_n may be given an orthogonal Young–Yamanouchi realisation (cf Robinson 1961, p 38). Thus the ordinary irreps of S_n are all of the orthogonal type, though not necessarily unimodular. A simple prescription for determining whether an irrep of S_n is unimodular or not has been given by King (1974), and readily allows us to assert that the $[n-1, 1]$ irrep of S_n is never unimodular. We note however that the sum of an even number of non-unimodular orthogonal irreps will always form a reducible unimodular representation of S_n.

The product of two ordinary characters of S_n may be expressed as a sum of simple ordinary characters of S_n by standard use of the character tables (cf Ledermann 1977). If (λ) and (μ) are partitions of the same integer n then

$$\chi_\rho^{(\lambda)}\chi_\rho^{(\mu)} = g_{\lambda\mu}{}^{\nu}\chi_\rho^{(\nu)} \tag{1}$$

and use of the character table yields the integers $g_{\lambda\mu}{}^{\nu}$. However, the object of this paper is to be able to evaluate Kronecker products of the irreps of S_n without explicit use of the character tables.

It is well known that the properties of the ordinary characters of S_n may be expressed in terms of those of the Schur functions (or S-functions) (cf Ledermann 1977, Littlewood 1950). The outer multiplication of S-functions of weights n and m, $\{\lambda\}\cdot\{\mu\}$, is evaluated by the Littlewood–Richardson rule (cf Littlewood 1950) and corresponds to the decomposition of the induced representation of S_{n+m} from $S_n\times S_m$. The inner product $\{\lambda\}\circ\{\mu\}$ of two S-functions of the same weight, say n, has been defined (Littlewood 1956) by the relation

$$\{\lambda\}\circ\{\mu\}=g_{\lambda\mu}{}^{\nu}\{\nu\} \tag{2}$$

where there is a one-to-one correspondence between the partitions and $g_{\lambda\mu}{}^{\nu}$ appearing in (1) and (2). The systematic evaluation of the inner products of S-functions has been the subject of many investigations (cf Littlewood 1956, Robinson 1961, Butler and Wybourne 1969). The inner products of S-functions may be systematically evaluated without explicit use of character tables, and thus the Kronecker products for any S_n may be resolved. However, such an approach inherently yields n-dependent information and as such frequently obscures underlying simplicities in the theory.

The possibility of developing an essentially n-independent resolution of the Kronecker products was first considered by Murnaghan (1937, 1938), who suggested the use of a 'reduced' notation for labelling the irreps of S_n that is n-independent. In the reduced notation, the irrep of S_n usually labelled by the symbol $[\lambda]\equiv[n-m,\mu_1,\mu_2,\ldots]$, with (μ) being a partition of m, is labelled by the symbol $\langle\mu\rangle\equiv\langle\mu_1,\mu_2,\ldots\rangle$. We shall use angular brackets $\langle\ \rangle$ to specify irreps of S_n in the reduced notation (Butler and King 1973). The reduced notation for the ordinary irreps of S_n arises naturally out of the embedding of the symmetric group in the linear group L_n (Littlewood 1958a). Since the ordinary irreps of S_n are orthogonal, including those of the defining irrep $\langle 1\rangle$, it is possible to treat S_n as a subgroup of O_n (Butler and King 1973), i.e.

$$L_n\downarrow O_n\downarrow S_n. \tag{3}$$

Indeed, for the ordinary irreps of S_n we may make the embedding $O_{n-1}\supset S_n$, an embedding exploited by Butler and King who have given extensive branching rules for $O_{n-1}\downarrow S_n$.

The use of the reduced notation has led to the n-independent evaluation of the Kronecker product of symmetric group representations $\langle\lambda\rangle$ and $\langle\mu\rangle$ as (Littlewood 1958a)

$$\langle\lambda\rangle\langle\mu\rangle=\sum_{\alpha,\beta,\gamma}\langle(\{\lambda\}/\{\alpha\}\{\beta\})\cdot(\{\mu\}/\{\alpha\}\{\gamma\})\cdot(\{\beta\}\circ\{\gamma\})\rangle \tag{4}$$

where $\{\beta\}$ and $\{\gamma\}$ are necessarily both partitions of the same number. It is important to note that in transforming from the n-independent reduced notation $\langle\mu\rangle$ to the n-dependent standard notation $[\lambda]=[n-m,(\mu)]$ the resulting symbols may not be in the standard form. However, non-standard symbols can always be reordered to give the standard form by noting that (Murnaghan 1938)

$$[\lambda_1,\ldots,\lambda_i,\lambda_{i+1},\ldots,\lambda_k]=-[\lambda_1,\ldots,\lambda_{i+1}-1,\lambda_i+1,\ldots,\lambda_k]. \tag{5}$$

The group S_n possesses two ordinary one-dimensional irreps, $[n]$ and $[1^n]$. Starting with an irrep $[\lambda]$ of S_n, we may form an irrep $[\tilde{\lambda}]$ by noting that

$$[\lambda][1^n]=[\tilde{\lambda}]. \tag{6}$$

The irrep $[\tilde{\lambda}]$ will be said to be the *associate* irrep of $[\lambda]$. If $[\lambda]\equiv[\tilde{\lambda}]$ the two irreps will be said to be *self-associated*; otherwise they are mutually associated. Self-associated irreps of S_n will be indicated by use of a dagger. Thus $[321]^{\dagger}$ designates a self-associated irrep while $[222]$ and $[\widetilde{222}]\equiv[3^2]$ are mutually associated irreps of S_6.

In the reduced notation $[n]$ and $[1^n]$ become labelled as $\langle 0\rangle$ and $\langle 1^{n-1}\rangle$. The $\langle 1^{n-1}\rangle$ irrep will frequently be designated as $\langle\tilde{0}\rangle$, where the tilde reminds us that it is formed from the association of $\langle 0\rangle$. Finally, we shall write

$$\langle\tilde{\mu}\rangle=\langle\tilde{0}\rangle\langle\mu\rangle \tag{7}$$

which is the n-independent version of (6). Care must be exercised in interpreting $\langle\tilde{\mu}\rangle$ in the reduced notation, as the tilde operation does not imply conjugation of the partition (λ) as it does in the standard notation. Thus $\langle 1\rangle\leftrightarrow[n-1,1]$ while $\langle\tilde{1}\rangle\leftrightarrow[\widetilde{n-1,1}]\equiv[21^{n-2}]\leftrightarrow\langle 1^{n-2}\rangle$.

We note that self-association for the ordinary irreps of S_n is an n-dependent property. The equivalence or otherwise of $\langle\mu\rangle$ and $\langle\tilde{\mu}\rangle$ can only be decided for particular values of n. Thus $\langle 21\rangle$ is self-associated for S_6 but not for any other value of n.

3. Spin representations of S_n

The symmetric group S_n of order $n!$ has two spin groups Γ_n and Γ'_n of order $2(n!)$ (Morris 1962a). It may be shown (Morris 1961) that only one of the spin groups needs to be considered since the characters of the two groups are trivially related. The characters of the positive classes of Γ_n and Γ'_n are the same, whilst the characters of the negative classes of Γ'_n are found by multiplying the corresponding character of Γ_n by $\mathrm{i}=\sqrt{-1}$. The group Γ_n is isomorphic to Schur's $\mathfrak{T}_n$ group (Schur 1911) and we shall restrict our attention to just the group Γ_n.

The two-valued spin irreps of S_n correspond to single-valued irreps of Γ_n and may be uniquely labelled by the ordered partitions of n into k unequal parts (Schur 1911), i.e.

$$\lambda_1>\lambda_2>\ldots>\lambda_k>0 \qquad \lambda_1+\lambda_2+\ldots+\lambda_k=n.$$

To distinguish spin irreps from ordinary irreps of S_n we shall use a prime. Thus $[421]'$ is a spin irrep while $[421]$ is an ordinary irrep of S_n.

If $(n-k)$ is *even* the irrep is self-associated and will be designated as $[\lambda]'^{\dagger}$ while if $(n-k)$ is *odd* we obtain an associated pair of spin irreps designated as $[\lambda]'$ and $[\tilde{\lambda}]'$. We note that $[\lambda]'+[\tilde{\lambda}]'$ corresponds to a reducible self-associated representation for $(n-k)$ odd. As a consequence we shall often use $[\lambda]'^{\dagger}$ without regard to the parity of $(n-k)$, with the understanding that if $(n-k)$ is odd then

$$[\lambda]'^{\dagger}\equiv[\lambda]'+[\tilde{\lambda}]' \qquad (n-k)\ \textit{odd}. \tag{8}$$

For each value of n there is a basic spin representation $[n]'^{\dagger}$ of degree $2^{[n/2]}$, where $[x]$ denotes the greatest integer less than or equal to x. The basic spin representation is such a representation from which every representation of S_n arises in a Kronecker power of the basic spin representation. Morris (1962a) has given a prescription for calculating the simple spin characters of Γ_n for the positive classes $(\pi)=(1^{\alpha_1}3^{\alpha_2}5^{\alpha_3}\ldots)$ based on the properties of Q-functions, and has gone on to show that for $(n-k)$ *odd* the negative class $(\lambda_1\lambda_2\ldots\lambda_k)$ has a non-zero spin character given by

$$\chi^{[\lambda]'}_{(\lambda)}=\mathrm{i}^{(n-k+1)/2}[\lambda_1\lambda_2\ldots\lambda_k/2]^{1/2} \tag{9}$$

with $\chi_{(\lambda)}^{[\tilde{\lambda}]'} = -\chi_{(\lambda)}^{[\lambda]'}$ while the spin character of every other negative class is zero. From (9) we conclude that a spin irrep of S_n will be necessarily *complex* if $(n-k+1)/2$ is *odd*.

In dealing with associated spin representations it is useful to exploit the properties of difference characters. Following the notation given by (8), we have

$$[\lambda]'^{\dagger} = [\lambda]' + [\tilde{\lambda}]' \qquad (n-k)\ odd \tag{10}$$

and the *difference*

$$[\lambda]'' = [\lambda]' - [\tilde{\lambda}]' \qquad (n-k)\ odd. \tag{11}$$

Hence

$$[\lambda]' = ([\lambda]'^{\dagger} + [\lambda]'')/2 \tag{12a}$$

$$[\tilde{\lambda}]' = ([\lambda]'^{\dagger} - [\lambda]'')/2. \tag{12b}$$

For the negative class $(\lambda_1\lambda_2\ldots\lambda_k)$ we have the difference character

$$\chi_{(\lambda)}^{[\lambda]''} = \mathrm{i}^{(n-k+1)/2}(2\lambda_1\lambda_2\ldots\lambda_k)^{1/2}. \tag{13}$$

The difference character for all other classes is zero.

4. Properties of associated and self-associated representations

In the previous sections we have noted that both the ordinary and spin irreps of S_n may be divided into two classes: associated and self-associated irreps. This is a general property of groups that contain the one-dimensional alternating irrep.

Two theorems concerning associated and self-associated irreps play an important role in our subsequent analysis of the properties of the irreps of S_n. The proofs of these theorems follow trivially from the definitions of the self-associated and associated irreps†. These theorems reinforce the usefulness of the notation developed in (7) and (10).

Theorem 1. If a group G contains a subgroup H with $\lambda_i^{\dagger}$, λ_i, $\tilde{\lambda}_i(\equiv \epsilon_G\lambda_i)$ and $\rho_i^{\dagger}$, ρ_i, $\tilde{\rho}_i(\equiv \epsilon_H\rho_i)$ being their respective self-associated and associated irreps and ϵ_G, ϵ_H their corresponding alternating irreps, then

(i) $$\lambda_i^{\dagger}\downarrow H = a_i^j\rho_j^{\dagger} + b_i^j(\rho_j + \tilde{\rho}_j) \tag{14a}$$

(ii) $$\text{if } \lambda_i\downarrow H = a_i^j\rho_j^{\dagger} + b_i^j\rho_j \quad \text{then } \tilde{\lambda}\downarrow H = a_i^j\rho_j^{\dagger} + b_i^j\tilde{\rho}_j \tag{14b}$$

where the coefficients a_i^j, b_i^j are non-negative integer multiplicity numbers and the Einstein summation convention is adopted.

Theorem 2. Let $\lambda_i^{\dagger}$, λ_i, $\tilde{\lambda}_i(\equiv \epsilon_G\lambda_i)$ be self-associated and associated irreps of a group G. The Kronecker products of the irreps of G necessarily satisfy the identities

(i) $$\lambda_i^{\dagger}\times\lambda_j = \lambda_i^{\dagger}\times\tilde{\lambda}_j = a_{ij}^k\lambda_k^{\dagger} + b_{ij}^k(\lambda_k + \tilde{\lambda}_k) \tag{15a}$$

(ii) $$\lambda_i^{\dagger}\times\lambda_j^{\dagger} = a_{ij}^k\lambda_k^{\dagger} + b_{ij}^k(\lambda_k + \tilde{\lambda}_k) \tag{15b}$$

† e.g. if $\lambda^{\dagger}\downarrow H = a_i^j\rho_j^{\dagger} + b_i^j\rho_j + c_i^j\tilde{\rho}_j$ then the self-association property leads directly to $b_i^j = c_i^j$ etc.

(iii) $\tilde{\lambda}_i \times \tilde{\lambda}_j = \lambda_i \times \lambda_j$ (15c)

(iv) if $\lambda_i \times \lambda_j = a_{ij}^k \lambda_k^\dagger + b_{ij}^k \lambda_k$
then $\lambda_i \times \tilde{\lambda}_j = a_{ij}^k \lambda_k^\dagger + b_{ij}^k \tilde{\lambda}_k$. (15d)

5. S_n as a subgroup of O_n

As noted earlier, since the ordinary irreps of S_n are orthogonal, including that of the defining irrep $\langle 1\rangle$, it is possible to treat S_n as a subgroup of O_n. Two embeddings deserve consideration. The first is the non-unimodular embedding defined by

$$[1]\downarrow\langle 1\rangle+\langle 0\rangle \tag{16}$$

which is possible for $O_n \supset S_n$ but not for $SO_n \supset S_n$. The second is the unimodular embedding defined by

$$[1]\downarrow\langle 1\rangle+\langle \tilde{0}\rangle \tag{17}$$

which makes possible the embedding of S_n in SO_n. However, for *n even* the basic spin irrep is non-unimodular and hence cannot admit a proper embedding in SO_n. Both embeddings are possible for $O_n \supset S_n$, and for our purposes we will restrict our attention to the non-unimodular embedding (16).

We note that for the ordinary irreps of S_n we can make an embedding in O_{n-1} (Butler and King 1973). However, such an embedding cannot be maintained for the spin irreps of S_n.

Given the embedding (16), the decomposition of an arbitrary ordinary irrep $[\lambda]$ of O_n into S_n irreps will yield the terms contained in the plethysm

$$(\langle 1\rangle+\langle 0\rangle)\otimes[\lambda]. \tag{18}$$

This plethysm can be converted into a plethysm involving S-functions by writing (Butler and King 1973)

$$[\lambda]=\{\lambda\}\Big/\sum_r ((-\{2\})\otimes\{r\}) \tag{19}$$

leading to the evaluation of plethysms of the type

$$(\langle 1\rangle+\langle 0\rangle)\otimes\{\mu\}. \tag{20}$$

The evaluation of these plethysms can be made by first noting that (Littlewood 1958a)

$$(\langle 1\rangle+\langle 0\rangle)\otimes\{1^r\}=\langle 1^r\rangle+\langle 1^{r-1}\rangle. \tag{21}$$

Furthermore, any S-function $\{\mu\}$ may be expanded as the product of S-functions of the type $\{1^x\}$ by noting that (Littlewood 1950)

$$\{1^x\}\equiv a_x, \tag{22}$$

where a_x is an elementary symmetric function $\Sigma\, \alpha_1\alpha_2 \ldots \alpha_x$, and that

$$\{\mu\}=|a_{\tilde{\mu}_s-s+t}| \tag{23}$$

where $(\tilde{\mu})$ is the partition conjugate to (μ). Thus it becomes possible to rewrite (20) as

$$(\langle 1\rangle+\langle 0\rangle)\otimes\{\mu\}=(\langle 1\rangle+\langle 0\rangle)\otimes(|a_{\tilde{\mu}_s-s+t}|). \tag{24}$$

The resulting plethysms may then be evaluated by noting that

$$(\langle 1\rangle+\langle 0\rangle)\otimes a_i a_j \ldots a_x = (\langle 1^i\rangle+\langle 1^{i-1}\rangle)(\langle 1^j\rangle+\langle 1^{j-1}\rangle)\ldots(\langle 1^x\rangle+\langle 1^{x-1}\rangle) \tag{25}$$

with the Kronecker products being evaluated using (4).

The above prescription, while tedious in application, is complete and capable of being programmed for computer evaluation. As an example we list the $O_n\downarrow S_n$ branching rules for partitions of four or less in table 1.

Table 1. Branching rules $O_n \to S_n$.

O_n	S_n
$[0]$	$\langle 0\rangle$
$[1]$	$\langle 1\rangle+\langle 0\rangle$
$[1^2]$	$\langle 1^2\rangle+\langle 1\rangle$
$[2]$	$\langle 2\rangle+2\langle 1\rangle+\langle 0\rangle$
$[1^3]$	$\langle 1^3\rangle+\langle 1^2\rangle$
$[21]$	$\langle 21\rangle+2\langle 2\rangle+2\langle 1^2\rangle+2\langle 1\rangle$
$[3]$	$\langle 3\rangle+2\langle 2\rangle+\langle 1^2\rangle+3\langle 1\rangle+2\langle 0\rangle$
$[1^4]$	$\langle 1^4\rangle+\langle 1^3\rangle$
$[21^2]$	$\langle 21^2\rangle+2\langle 21\rangle+2\langle 1^3\rangle+\langle 2\rangle+2\langle 1^2\rangle$
$[2^2]$	$\langle 2^2\rangle+2\langle 21\rangle+\langle 3\rangle+3\langle 2\rangle+\langle 1^2\rangle+\langle 1\rangle$
$[31]$	$\langle 31\rangle+3\langle 21\rangle+2\langle 3\rangle+\langle 1^3\rangle+4\langle 2\rangle+5\langle 1^2\rangle+4\langle 1\rangle+\langle 0\rangle$
$[4]$	$\langle 4\rangle+\langle 21\rangle+2\langle 3\rangle+4\langle 2\rangle+2\langle 1^2\rangle+5\langle 1\rangle+3\langle 0\rangle$

We must now consider the spin irreps of $O_n \supset S_n$. The spin irreps of O_n may be labelled as $[\Delta;\lambda]$ with the basic spin irrep being designated by $\Delta\equiv[\Delta;0]$ (cf King 1975). It is readily seen that under $O_n\downarrow S_n$ we have

$$\Delta\downarrow[n]'^{\dagger}. \tag{26}$$

This result holds for all n if we remember the notation introduced in (8). This suggests that it should be possible to develop an n-independent reduced notation for the spin irreps as well as for the ordinary irreps of S_n, i.e.

$$\Delta\downarrow\langle 0\rangle'^{\dagger}. \tag{27}$$

We shall proceed to develop, and exploit, just such a reduced notation for spin irreps of S_n.

The spin character of the irrep $[\Delta;\lambda]$ of O_n may be expressed as the product of the basic spin irrep Δ with a sum over the ordinary irreps of O_n by writing (King 1975)

$$[\Delta;\lambda]=\Delta\sum_m(-1)^m[\lambda/m]. \tag{28}$$

The reduction to S_n may now be accomplished by using (26) to reduce the basic spin irrep and (18) to reduce the ordinary irreps of O_n into irreps of S_n, leading finally to

$$[\Delta;\lambda]\downarrow\langle 0\rangle'^{\dagger}\sum\langle\pi\rangle. \tag{29}$$

The right-hand side is a compound spin character of S_n which has been expressed in an n-independent notation and must be expressible in terms of simple spin characters in an n-independent manner. To this end we introduce the reduced notation

$$\langle\mu\rangle'=[n-m,(\mu)]' \tag{30}$$

where (μ) is a partition of m. We then rewrite (29) as

$$[\Delta;\lambda]\downarrow\sum\langle\mu\rangle'. \tag{31}$$

A short list of the decompositions arising in the reduction of (29) is given in table 2. In order to develop our theory further, it is essential to be able to express an arbitrary spin irrep $\langle\mu\rangle'$ as a product of the basic spin irrep $\langle 0\rangle'^{\dagger}$ with a sum over the ordinary irreps of S_n and vice versa.

Having established the validity of the reduced notation for spin irreps of S_n, we next consider the n-dependence of the dimensional formulae for the ordinary and spin irreps of S_n.

Table 2. $[\Delta;0]\downarrow\langle 0\rangle'^{\dagger}\,\Sigma\,\langle\pi\rangle$.

$[\Delta;0]$	$\langle 0\rangle'^{\dagger}\langle 0\rangle$
$[\Delta;1]$	$\langle 0\rangle'^{\dagger}\langle 1\rangle$
$[\Delta;1^2]$	$\langle 0\rangle'^{\dagger}(\langle 1^2\rangle-\langle 0\rangle)$
$[\Delta;2]$	$\langle 0\rangle'^{\dagger}(\langle 2\rangle+\langle 1\rangle+\langle 0\rangle)$
$[\Delta;1^3]$	$\langle 0\rangle'^{\dagger}(\langle 1^3\rangle-\langle 1^2\rangle)$
$[\Delta;21]$	$\langle 0\rangle'^{\dagger}(\langle 21\rangle+\langle 2\rangle+\langle 1^2\rangle)$
$[\Delta;3]$	$\langle 0\rangle'^{\dagger}(\langle 3\rangle+\langle 2\rangle+\langle 1^2\rangle+2\langle 1\rangle+\langle 0\rangle)$
$[\Delta;1^4]$	$\langle 0\rangle'^{\dagger}(\langle 1^4\rangle-\langle 1^3\rangle)$
$[\Delta;21^2]$	$\langle 0\rangle'^{\dagger}(\langle 21^2\rangle+\langle 21\rangle+\langle 1^3\rangle-\langle 2\rangle-\langle 1\rangle)$
$[\Delta;2^2]$	$\langle 0\rangle'^{\dagger}(\langle 2^2\rangle+\langle 21\rangle+\langle 3\rangle+2\langle 2\rangle-\langle 1^2\rangle+\langle 1\rangle+\langle 0\rangle)$
$[\Delta;31]$	$\langle 0\rangle'^{\dagger}(\langle 31\rangle+2\langle 21\rangle+\langle 2\rangle+\langle 1^3\rangle+\langle 2\rangle+3\langle 1^2\rangle\langle 1\rangle-\langle 0\rangle)$
$[\Delta;4]$	$\langle 0\rangle'^{\dagger}(\langle 4\rangle+\langle 21\rangle+\langle 3\rangle+3\langle 2\rangle+\langle 1^2\rangle+3\langle 1\rangle+2\langle 0\rangle)$

6. Dimensions of irreps of S_n

The formula for the dimensions $f^{[\lambda]}$ of the ordinary irreps $[\lambda]$ of S_n is well known (Robinson 1961):

$$f^{[\lambda]}=n!/H(\lambda) \tag{32}$$

where the hook length factor is given by

$$H(\lambda)=\prod_{ij}(\lambda_i-j+\tilde{\lambda}_j-i+1) \tag{33}$$

where (λ) and $(\tilde{\lambda})$ are mutually conjugate partitions of n. It is possible to display the n-dependence of an irrep $[\lambda]$ explicitly, taking advantage of the reduced notation to give (Butler and King 1973)

$$f^{[\lambda]}=f_n^{\langle\mu\rangle}=\frac{1}{H(\mu)}\prod_{i=1}^{r}(n-m-\mu_i+i) \tag{34}$$

where now $H(\mu)$ is an n-independent function and the remaining product term is explicitly n-dependent.

The dimension formula for spin irreps of S_n was given long ago (Schur 1911) for a k-part partition $[\lambda_1\ldots\lambda_k]$ as

$$f^{[\lambda]'}=2^{[(n-k)/2]}n!\prod_{i=1}^{k}(\lambda_i!)^{-1}\prod_{1\leqslant l<s\leqslant k}\left(\frac{\lambda_l-\lambda_s}{\lambda_l+\lambda_s}\right). \tag{35}$$

Taking advantage of the reduced notation, we find

$$f_n^{\langle\mu\rangle'} = 2^{[(n-r-1)/2]} C_m^n \prod_i^r \left(\frac{n-m-u_i}{n-m+u_i}\right) f^{\langle\mu\rangle'} \tag{36}$$

where r is the number of parts and m the weight of the partition (μ), C_m^n a binomial coefficient and

$$f^{\langle\mu\rangle'} = m! \prod_{i=1}^{k} (\mu_i!)^{-1} \prod_{1\leq l<s\leq r} \left(\frac{\mu_l-\mu_s}{\mu_l+\mu_s}\right) \tag{37}$$

is an n-independent factor which we shall term the reduced dimension of the spin irreps of S_n.

The above results allow us to determine explicitly the n-dependence of the dimensional formula appropriate to any reduced spin irrep $\langle\mu\rangle'$. Thus for $\langle 31\rangle'$ we readily deduce that $f^{\langle 31\rangle'} = 2$ and thence

$$f^{[n-4,31]'} = f_n^{\langle 31\rangle'} = 2^{[(n-3)/2]} n(n-2)(n-5)(n-7)/12.$$

7. Q-functions and S_n irreps

The S-functions were introduced by Schur in his development of the theory of the ordinary irreps of S_n (Schur 1901). Later, in his study of the linear fractional substitution representation group $\mathfrak{T}_n$ of S_n, he introduced a second symmetric function termed the Q-function (Schur 1911, p 224). It was much later realised that the S- and Q-functions were particular cases of what are now known as Hall–Littlewood functions (Hall 1957, Littlewood 1961). The properties of the Hall–Littlewood functions have been surveyed by Morris (1976) and a concise description has been given by Thomas (1976).

Consider a symmetric function of the indeterminants $\alpha_1, \alpha_2, \ldots, \alpha_n$; then if

$$\prod \frac{1}{(1-\alpha_i x)} = 1 + \sum_{r=1}^{\infty} h_r x^r \tag{38}$$

then h_r defines the Schur functions $\{r\}$. The symmetric function h_r is the sum of all monomial symmetric functions of degree r in the α_i's. The Schur function $\{\lambda\}$ associated with an arbitrary partition (λ) is then defined by (Littlewood 1950)

$$\{\lambda\}_h = |h_{\lambda_i - i + j}|. \tag{39}$$

A Schur function $\{\lambda\}$ may be conveniently expanded into products of h_k by use of the Young *raising operator* δ_{ij}, which operates on a partition (λ) by increasing λ_i by one and decreasing λ_j by one with $i<j$. We then have (Thomas 1980)

$$\{\lambda\}_h = \prod_{i<j} (1-\delta_{ij}) h_\lambda \qquad \text{where } h_\lambda \equiv h_{\lambda_1} h_{\lambda_2} \ldots \tag{40}$$

and conversely

$$h_\lambda = \prod_{i<j} \frac{1}{(1-\delta_{ij})} \{\lambda\}_h \tag{41}$$

remembering that

$$1/(1-\delta_{ij}) = 1+\delta_{ij}+\delta_{ij}^2+\dots. \tag{42}$$

As usual, non-standard symbols are reordered to standard form using (5).

The Schur functions may be generalised by considering the expressions

$$\prod \frac{(1-t\alpha_i x)}{(1-s\alpha_i x)} = 1+\sum_{r=1}^{\infty} q_r x^r \tag{43}$$

with

$$\{\lambda\}_q = |q_{\lambda_i - i+j}| = \prod_{i<j}(1-\delta_{ij})q_\lambda. \tag{44}$$

It is readily seen that $t=0$, $s=1$ yields the usual S-functions.

A further generalisation is made possible by considering two sets of indeterminants $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\beta_1, \beta_2, \dots, \beta_n$ and the function

$$\prod_{i,j} \frac{(1-t\alpha_i\beta_j x)}{(1-\alpha_i\beta_j x)} = 1+\sum_{r=1}^{\infty} P_r x^r \tag{45}$$

together with the requirement that P_r is expressed in the form

$$P_r = \sum_\lambda k_\lambda(t) Q_\lambda(t) Q'_\lambda(t) \tag{46}$$

where $Q_\lambda(t)$ is a symmetric function in $\alpha_1, \alpha_2, \dots, \alpha_n$, $Q'_\lambda(t)$ is the same symmetric function but in $\beta_1, \beta_2, \dots, \beta_n$, and $k_\lambda(t)$ is a polynomial in t which depends on the partition (λ). The functions $Q_\lambda(t)$ are referred to as Hall–Littlewood functions.

The Young raising operators δ_{ij} may be used to express the S-functions $\{\lambda\}_q$ in terms of the Hall–Littlewood functions to give (Littlewood 1961, Thomas 1976)

$$\{\lambda\}_q = \prod_{i<j}(1-t\delta_{ij})Q_\lambda(t) \tag{47}$$

and vice versa,

$$\begin{aligned} Q_\lambda(t) &= \prod_{i<j} \frac{1}{(1-t\delta_{ij})}\{\lambda\}_q \\ &= \prod_{i<j}(1+t\delta_{ij}+t^2\delta_{ij}^2+\dots)\{\lambda\}_q. \end{aligned} \tag{48}$$

In the theory of the symmetric group, the functions with $t=0$ are simply the Schur functions that arise in the theory of the ordinary irreps of S_n. Specifically, if (λ) is a partition of n and $\chi_\rho^{[\lambda]}$ is the irreducible character of the irrep $[\lambda]$ for the cycle $\rho = (1^{\rho_1}2^{\rho_2}\dots n^{\rho_n})$, then

$$\{\lambda\}_h = \frac{1}{n!}\sum_\rho g_\rho \chi_{(\rho)}^{[\lambda]} S_\rho \tag{49}$$

where g_ρ is the order of the class (ρ) and S_ρ is the symmetric power sum.

The Hall–Littlewood functions $Q_\lambda(-1)$ are identical to the Q-functions introduced by Schur (1911), and play a similar role for the spin characters of S_n as do the S-functions for the ordinary characters. Henceforth we shall write $Q_{(\lambda)} = Q_\lambda(-1)$ and refer to the $Q_{(\lambda)}$ simply as Q-functions. Each Q-function will be associated with a partition (λ) of n into k integer parts. The partition need not be in standard form.

Q-functions corresponding to non-standard partitions may be converted into the standard descending order by noting the following four rules (Morris 1962a, 1976).

(1) If any two parts are equal the Q-function is zero.

(2) $$Q_{(\ldots,\lambda_i,\lambda_{i+1},\ldots)} = -Q_{(\ldots,\lambda_{i+1},\lambda_i,\ldots)} \tag{50}$$

and hence the Q-function is zero if any two parts are equal.

(3) A Q-function will be zero if any part is negative and the magnitude of every part is different.

(4) $$Q_{(\ldots,\lambda_i,-\lambda_i,\ldots)} = 0 \tag{51}$$

while

$$Q_{(\ldots,-\lambda_i,\lambda_i,\ldots)} = 2(-1)^{\lambda_i} Q_{(\ldots,\lambda_{i-1},\lambda_{i+1},\ldots)}. \tag{52}$$

Application of the above rules allows us to reduce any Q-function either to zero or to the form $Q_{(\lambda)}$ where (λ) is a partition of n into k unequal parts such that

$$\lambda_1 > \lambda_2 > \ldots > \lambda_k > 0, \tag{53}$$

the same condition that applies for the existence of spin irreps of S_n.

The connection between Q-functions and the spin characters of S_n is made explicit by Schur's relation (Schur 1911)

$$Q_{(\lambda)} = 2^{(k+p+\epsilon)/2} \sum_{(\pi)} \frac{h_\pi}{h} \zeta_\pi^{[\lambda]'} S_\pi \tag{54}$$

where $\zeta_\pi^{[\lambda]'}$ is a simple spin character of the class $(\pi) = (1^{\alpha_1} 3^{\alpha_3} \ldots)$ involving odd cycles only, $p = \alpha_1 + \alpha_3 + \ldots$, h_π is the order of the class (π), h the order of Γ_n, $S_\pi = S_1^{\alpha_1} S_3^{\alpha_3} \ldots$ and $\epsilon = 0$ or 1 according as $(n-k)$ is even or odd. If $\epsilon = 0$ then $\zeta_{(\pi)}^{[\lambda]'}$ is a self-associated double spin character of S_n, and if $\epsilon = 1$ it is an associated spin character with $\zeta_{(\pi)}^{[\bar{\lambda}]'} = -\zeta_{(\pi)}^{[\lambda]'}$. Equation (54) may be contrasted with the corresponding result for S-functions, equation (49). In the latter case the summation is over all the classes of S_n, whereas in the former case the summation is restricted to the classes involving odd cycles only.

The outer product of two Q-functions, say $Q_{(\lambda)}$ and $Q_{(\mu)}$, of weights n and m may be resolved into a sum of Q-functions of weight $m+n$ to give

$$Q_{(\lambda)} \cdot Q_{(\mu)} = \Gamma_{\lambda\mu}{}^{\nu} Q_{(\nu)}. \tag{55}$$

The non-negative numbers $\Gamma_{\lambda\mu}{}^{\nu}$ may be determined by use of (48) to expand each Q-function as a sum of S-functions, then the outer products of the S-functions calculated by the usual Littlewood–Richardson rule, and then the resulting S-functions converted back into Q-functions using (47). Alternative methods are available (Morris 1962a, 1963, 1964a, b).

Q-function division may be defined by

$$Q_{(\lambda/\mu)} = \Gamma_{\mu}{}^{\nu}{}_{\lambda} Q_{(\nu)} \tag{56}$$

where $\Gamma_{\mu}{}^{\nu}{}_{\lambda}$ is the same as the coefficient that appears in the outer product

$$Q_{(\mu)} \cdot Q_{(\nu)} = \Gamma_{\mu\nu}{}^{\lambda} Q_{(\lambda)}. \tag{57}$$

Outer Q-function products are closely related to the induction $S_n \times S_m \uparrow S_{n+m}$ and the Q-function division to the restriction $S_{n+m} \downarrow S_n \times S_m$ for spin characters.

For later use we note that (cf Morris 1962a)

$$Q_{(\lambda)}Q_{(1)} = 2Q_{(\lambda_1+1,\ldots,\lambda_k)} + \ldots + 2Q_{(\lambda_1,\ldots,\lambda_k+1)} + Q_{(\lambda_1,\ldots,\lambda_k,1)} \tag{58}$$

and hence

$$Q_{(\lambda/1)} = 2[Q_{(\lambda_1-1,\ldots,\lambda_k)} + \ldots + Q_{(\lambda_1,\ldots,\lambda_{k-1})}] - \delta_{\lambda_k,1}Q_{(\lambda_1,\ldots,\lambda_k-1)}. \tag{59}$$

It is also useful to define the inner product of a Q-function with an S-function defined on the same indeterminants by writing

$$Q_{(\lambda)} \circ \{\mu\}_h = g_{\lambda\mu}{}^{\nu}Q_{(\nu)} \tag{60}$$

where (λ), (μ) and (ν) are all partitions of n. The inner product may be evaluated by using (48) to express $Q_{(\lambda)}$ as a sum of S-functions $\{\lambda\}_q$, then evaluating the S-function inner products and finally converting the resultant S-functions back into Q-functions using (47). In making use of (47) and (48), it is essential to realise that the Young raising operators must be used prior to use of the reordering rules.

8. The $\mathrm{S}_n \downarrow \mathrm{S}_{n-1}$ branching rule

The $\mathrm{S}_n \downarrow \mathrm{S}_{n-1}$ branching rule for ordinary irreps of S_n is well known, and in the reduced notation amounts to

$$\langle\mu\rangle \downarrow \langle\mu\rangle + \langle\mu/1\rangle \tag{61}$$

where $\langle\mu/1\rangle$ is determined by removing one cell from the Young diagram of (μ) in all ways that result in a standard Young diagram. The rule as stated (61) is n-independent.

The statement of a similar rule for the spin irreps of S_n is complicated by the existence of self-associated irreps and associated pairs of irreps. Two special cases have been discussed (Wales 1979), but no general statement of the rule for spin irreps appears to have been given. The general rule follows by noting (59) and theorem 1 of § 4 to give in the reduced notation

$$\langle\mu\rangle' \downarrow \langle\mu\rangle'^{\dagger} + \langle\mu/1\rangle'^{\dagger} - \delta_{\lambda_r,1}\langle\mu_1, \ldots, \mu_{r-1}\rangle' \tag{62a}$$

$$\langle\mu\rangle'^{\dagger} \downarrow \langle\mu\rangle'^{\dagger} + \langle\mu/1\rangle'^{\dagger} \tag{62b}$$

where $\langle\mu/1\rangle'$ is evaluated exactly as for the ordinary irreps and we use the notation implied by (8). In (62a) we clearly have $n-k$ *odd* whereas in (62b) $n-k$ is *even*.

The following examples illustrate the application of the two rules.

$$\langle 421\rangle' \downarrow \langle 421\rangle'^{\dagger} + \langle 42\rangle'^{\dagger} + \langle 321\rangle'^{\dagger} - \langle\widetilde{42}\rangle' \equiv \langle 421\rangle'^{\dagger} + \langle 42\rangle' + \langle 321\rangle'^{\dagger}.$$

Thus for $\mathrm{S}_{13} \downarrow \mathrm{S}_{12}$

$$[6421]' \downarrow [5421]'^{\dagger} + [642]' + [6321]'^{\dagger}.$$

Likewise

$$\langle 421\rangle'^{\dagger} \downarrow \langle 421\rangle'^{\dagger} + \langle 42\rangle'^{\dagger} + \langle 321\rangle'^{\dagger} \equiv \langle 421\rangle' + \langle\widetilde{421}\rangle + \langle 42\rangle'^{\dagger} + \langle 321\rangle' + \langle\widetilde{321}\rangle'.$$

Thus for $\mathrm{S}_{14} \downarrow \mathrm{S}_{13}$

$$[7421]'^{\dagger} \downarrow [6421]' + [\widetilde{6421}]' + [742]'^{\dagger} + [7321]' + [\widetilde{7321}]'.$$

9. Kronecker products of basic spin with ordinary irreps for S_n

We now consider the resolution of the compound character $\langle 0\rangle'\langle\pi\rangle$ as a sum of simple spin characters of S_n. This is entirely equivalent to resolving the Kronecker product of the basic spin character $\langle 0\rangle'$ with an ordinary character $\langle\pi\rangle$. The properties of Q-functions make this a comparatively easy task.

From (60), after noting (48), we have

$$Q_{(n)} \circ \{\mu\}_h = \{\mu\}_q = \prod_{i<j} (1+\delta_{ij}) Q_{(\mu)}. \tag{63}$$

In order to facilitate the reduced notation, we now introduce a special Young raising operator δ_{0j}, which in the reduced notation has the effect of decreasing μ_j by one unit. We can now write a reduced version of (47) and (48) appropriate to the Q-functions for any S_n as

$$\langle\mu\rangle_q = \prod_{0\leqslant i<j} (1+\delta_{ij}) Q_{\langle\mu\rangle} \tag{64}$$

$$Q_{\langle\mu\rangle} = \prod_{0\leqslant i<j} [1-\delta_{ij}+\delta_{ij}^2 \ldots]\langle\mu\rangle_q. \tag{65}$$

Equation (63) now becomes, in the reduced notation,

$$Q_{\langle 0\rangle} \circ \langle\mu\rangle_h = \prod_{0\leqslant i<j} (1+\delta_{ij}) Q_{\langle\mu\rangle}. \tag{66}$$

Application of (66) to an r-part reduced partition will yield $2^{C_2^{r+1}}$ terms, not necessarily all distinct. The results for $r \leqslant 3$ are given in table 3. For a specific partition, non-standard Q-functions may arise and must be reduced to the standard descending order by use of the modification rules given earlier.

Thus in the case of a four-part reduced partition we expect (66) to yield 1024 terms. Specialisation to the reduced partition $\langle 4321\rangle$ results in the survival of just 88 Q-functions, and of these only 25 are distinct. Restriction to a particular value of n may result in even fewer terms surviving. Thus if $n = 15$ the product involving $\langle 4321\rangle$ yields a total of 56 Q-functions of which 15 are distinct.

The evaluation of a specific inner product, say

$$Q_{[n]} \circ [\lambda] = g_{n\lambda}{}^{\mu} Q_{[\mu]},$$

may be checked by noting that

$$f^{[n]'+} f^{[\lambda]} = 2^{([k-n(\mathrm{mod}\ 2))/2]} g_{n\lambda}{}^{\mu} f^{[\mu]'+} \tag{67}$$

Table 3. $Q_{\langle 0\rangle} \circ \langle\lambda\rangle_h$ inner products.

$$Q_{\langle 0\rangle} \circ \langle p\rangle_h = Q_{\langle p\rangle} + Q_{\langle p-1\rangle}$$

$$Q_{\langle 0\rangle} \circ \langle pq\rangle_h = Q_{\langle pq\rangle} + 2Q_{\langle pq-1\rangle} + Q_{\langle pq-2\rangle} + Q_{\langle p+1,q-1\rangle} + Q_{\langle p+1,q-2\rangle} + Q_{\langle p-1q\rangle} + Q_{\langle p-1q-1\rangle}$$

$$\begin{aligned} Q_{\langle 0\rangle} \circ \langle pqr\rangle_h = {} & Q_{\langle pqr\rangle} + 4Q_{\langle pqr-1\rangle} + 4Q_{\langle pqr-2\rangle} + Q_{\langle pqr-3\rangle} + 2Q_{\langle pq-1r\rangle} + 4Q_{\langle pq-1r-1\rangle} \\ & + 2Q_{\langle pq-1r-2\rangle} + Q_{\langle pq-2r\rangle} + Q_{\langle pq-2r-1\rangle} + Q_{\langle pq+1r-1\rangle} + 2Q_{\langle pq+1r-2\rangle} \\ & + Q_{\langle pq+1r-3\rangle} + Q_{\langle p-1qr\rangle} + 2Q_{\langle p-1qr-1\rangle} + Q_{\langle p-1qr-2\rangle} + Q_{\langle p-1q-1r\rangle} \\ & + Q_{\langle p-1q-1r-1\rangle} + Q_{\langle p-1q+1r-1\rangle} + Q_{\langle p-1q+1r-2\rangle} + 2Q_{\langle p+1qr-1\rangle} + 4Q_{\langle p+1qr-2\rangle} \\ & + 2Q_{\langle p+1qr-3\rangle} + Q_{\langle p+1q-1r\rangle} + 4Q_{\langle p+1q-1r-1\rangle} + 4Q_{\langle p+1q-1r-2\rangle} \\ & + Q_{\langle p+1q-1r-3\rangle} + Q_{\langle p+1q-2r\rangle} + 2Q_{\langle p+1q-2r-1\rangle} + Q_{\langle p+1q-2r-2\rangle} \\ & + Q_{\langle p+1q+1r-2\rangle} + Q_{\langle p+1q+1r-3\rangle} + Q_{\langle p+2qr-3\rangle} + Q_{\langle p+2q-1r-1\rangle} \\ & + 2Q_{\langle p+2q-1r-2\rangle} + Q_{\langle p+2q-1r-3\rangle} + Q_{\langle p+2q-2r-1\rangle} + Q_{\langle p+2q-2r-2\rangle} \end{aligned}$$

where k is the number of parts in the partition (μ), $g_{n\lambda}{}^{\mu}$ is the multiplicity associated with $Q_{[\mu]}$ and the dimensional factors are evaluated using (32) for the ordinary irreps and (35) for the spin irreps.

As an example, we note from table 3 that for $\langle 321\rangle$ we have

$$\begin{aligned} Q_{\langle 0\rangle}\circ\langle 321\rangle = {} & Q_{\langle 5\rangle}+2Q_{\langle 4\rangle}+Q_{\langle 3\rangle}+Q_{\langle 51\rangle}+2Q_{\langle 42\rangle} \\ & +3Q_{\langle 32\rangle}+3Q_{\langle 41\rangle}+3Q_{\langle 31\rangle}+Q_{\langle 21\rangle}+Q_{\langle 321\rangle}. \end{aligned}$$

Specialisation to $n = 11$ gives

$$\begin{aligned} Q_{[11]}\circ[5321] = {} & (Q_{[65]}+2Q_{[74]}+Q_{[83]})+(2Q_{[542]}+3Q_{[632]}+2Q_{[641]}+3Q_{[731]} \\ & +Q_{[821]})+Q_{[5321]} \end{aligned}$$

and (67) yields

$$\begin{aligned} 32\times 2310 = {} & 1344+2\times 2880+2400+2\times(2\times 1760+3\times 2464+3\times 3168 \\ & +3\times 3168+1232)+2\times 1056 = 73\,920 \end{aligned}$$

whereas for $n = 10$ we find

$$Q_{[10]}\circ[4321] = (2Q_{[64]}+Q_{[73]})+3Q_{[532]}+2Q_{[541]}+3Q_{[631]}+Q_{[721]})+Q_{[4321]}$$

with (67) giving

$$\begin{aligned} 32\times 768 & = 2\times(2\times 672+768)+2\times(3\times 864+2\times 896+3\times 1600+800)+4\times 96 \\ & = 24\,576. \end{aligned}$$

We are now in a position to be able to give a complete algorithm to evaluate the Kronecker products $\langle 0\rangle'^{\dagger}\langle\mu\rangle$, as follows.

Algorithm 1.

(1) Evaluate the Q-function inner product $Q_{\langle 0\rangle}\circ\langle\mu\rangle_h$ using (66).

(2) Use the modification rules to convert any non-standard Q-function into standard descending order.

(3) Replace every Q-function, $Q_{\langle\rho\rangle}$, appearing in the product by

$$2^{[(k-n(\mathrm{mod}\,2))/2]}\langle\rho\rangle'^{\dagger}.$$

If $n-k$ is odd $\langle\rho\rangle'^{\dagger}\equiv\langle\rho\rangle'+\langle\tilde{\rho}\rangle'$.

Thus for $\langle 0\rangle'^{\dagger}\langle 321\rangle$ with *n odd* we obtain

$$\begin{aligned} \langle 0\rangle'^{\dagger}\langle 321\rangle = {} & \langle 5\rangle'^{\dagger}+2\langle 4\rangle'^{\dagger}+\langle 3\rangle'^{\dagger}+2(\langle 51\rangle'^{\dagger}+2\langle 42\rangle'^{\dagger}+3\langle 32\rangle'^{\dagger} \\ & +3\langle 41\rangle'^{\dagger}+3\langle 31\rangle'^{\dagger}+\langle 21\rangle'^{\dagger})+2\langle 321\rangle'^{\dagger} \end{aligned}$$

from which we deduce for $n = 11$

$$\begin{aligned} [11]'^{\dagger}[5321] = {} & [65]'^{\dagger}+2[74]'^{\dagger}+[83]'^{\dagger}+4[542]'^{\dagger}+6[632]'^{\dagger} \\ & +6[641]'^{\dagger}+6[731]'^{\dagger}+2[821]'^{\dagger}+2[5321]'^{\dagger}. \end{aligned}$$

Similarly for $n = 12$

$$\begin{aligned} [12]'^{\dagger}[6321]' = {} & 2([75]'^{\dagger}+2[84]'^{\dagger}+[93]'^{\dagger})+2([651]'^{\dagger}+2[642]'^{\dagger}+3[732]'^{\dagger} \\ & +3[741]'^{\dagger}+3[831]'^{\dagger}+[921]'^{\dagger})+4[6321]'^{\dagger}. \end{aligned}$$

We note that for *n even* $\langle 0\rangle'^{\dagger} = \langle 0\rangle' + \langle\tilde{0}\rangle'$ and our algorithm will not of course yield the product $\langle 0\rangle'\langle\mu\rangle$, except where $\langle\mu\rangle$ corresponds to a self-associated irrep. To separate out the terms $\langle 0\rangle'\langle\mu\rangle$ and $\langle\tilde{0}\rangle'\langle\mu\rangle$ from $\langle 0\rangle'^{\dagger}\langle\mu\rangle$ requires use of difference characters.

Our algorithm can be used to expand recursively any spin irrep $\langle\mu\rangle'^{\dagger}$ as a linear combination of terms of the type $\langle 0\rangle'^{\dagger}\langle\lambda\rangle$. For example, we may readily establish the results shown in table 4.

Table 4. Expansion of spin irreps† in terms of the basic spin irrep and ordinary irreps of S_n.

$$\langle 0\rangle'^{\dagger} = \langle 0\rangle'^{\dagger}\langle 0\rangle$$
$$\langle 1\rangle'^{\dagger} = (\tfrac{1}{2})\langle 0\rangle'^{\dagger}(\langle 1\rangle - \langle 0\rangle)$$
$$\langle 2\rangle'^{\dagger} = (\tfrac{1}{2})\langle 0\rangle'^{\dagger}(\langle 2\rangle - \langle 1\rangle + \langle 0\rangle)$$
$$\langle 21\rangle'^{\dagger} = \tfrac{1}{2}\langle 0\rangle'^{\dagger}(\langle 21\rangle - \langle 3\rangle - \langle 2\rangle)$$
$$\langle 3\rangle'^{\dagger} = (\tfrac{1}{2})\langle 0\rangle'^{\dagger}(\langle 3\rangle - \langle 2\rangle + \langle 1\rangle - \langle 0\rangle)$$
$$\langle 31\rangle'^{\dagger} = \tfrac{1}{2}\langle 0\rangle'^{\dagger}(\langle 31\rangle - \langle 21\rangle - \langle 4\rangle + \langle 2\rangle)$$
$$\langle 32\rangle'^{\dagger} = \tfrac{1}{2}\langle 0\rangle'^{\dagger}[\langle 32\rangle - \langle 41\rangle - \langle 31\rangle + \langle 5\rangle + \langle 4\rangle + \langle 3\rangle]$$
$$\langle 4\rangle'^{\dagger} = (\tfrac{1}{2})\langle 0\rangle'^{\dagger}[\langle 4\rangle - \langle 3\rangle + \langle 2\rangle - \langle 1\rangle + \langle 0\rangle]$$
$$\langle 41\rangle'^{\dagger} = \tfrac{1}{2}\langle 0\rangle'^{\dagger}[\langle 41\rangle - \langle 31\rangle + \langle 21\rangle - \langle 5\rangle - \langle 2\rangle]$$

† Where the coefficient appears as $(\frac{1}{2})$ it is to be included only for *n even*.

10. Kronecker products of spin with ordinary irreps

We now consider the general Kronecker product of a spin irrep $\langle\nu\rangle'^{\dagger}$ with an ordinary irrep $\langle\mu\rangle$, with the understanding that $\langle\nu\rangle'^{\dagger}$ is either a self-associated irrep or an associated pair of irreps. Algorithm 1 may be simply extended to give the following algorithm.

Algorithm 2.

(1) Expand the Q-function $Q_{\langle\nu\rangle}$ as a sum of S-functions using (48).

(2) Evaluate the terms in the relevant S-function inner products using (4).

(3) Express the resulting S-functions as Q-functions using (47).

(4) Use the modification rules to convert any non-standard Q-function into standard descending order.

(5) Replace every Q-function $Q_{\langle\rho\rangle}$, including $Q_{\langle\mu\rangle}$, by

$$2^{[(k-n(\mathrm{mod}\ 2))/2]}\langle\rho\rangle'^{\dagger}.$$

If $n-k$ is *odd* $\langle\rho\rangle'^{\dagger} \equiv \langle\rho\rangle' + \langle\tilde{\rho}\rangle'$.

The implementation of the above algorithm is seen in the evaluation of $\langle 2\rangle'\langle 1^2\rangle$ as follows.

$$Q_{\langle 2\rangle} \circ \langle 1^2\rangle_h = (\langle 2\rangle_q - \langle 1\rangle_q + \langle 0\rangle_q) \circ \langle 1^2\rangle_h$$
$$= \langle 21^2\rangle_q + \langle 31\rangle_q + \langle 21\rangle_q + \langle 3\rangle_q + 2\langle 1^2\rangle_q.$$

Use of (47) gives

$$\langle 21^2\rangle_q \to Q_{\langle 31\rangle} + Q_{\langle 21\rangle} + Q_{\langle 4\rangle} + 2Q_{\langle 3\rangle} + 2Q_{\langle 2\rangle} + Q_{\langle 1\rangle}$$

$$\langle 31\rangle_q \to Q_{\langle 31\rangle} + Q_{\langle 21\rangle} + Q_{\langle 4\rangle} + 2Q_{\langle 3\rangle} + Q_{\langle 2\rangle}$$

$$\langle 21\rangle_q \to Q_{\langle 21\rangle} + Q_{\langle 3\rangle} + 2Q_{\langle 2\rangle} + Q_{\langle 1\rangle}$$

$$\langle 3\rangle_q \to Q_{\langle 3\rangle} + Q_{\langle 2\rangle}$$

$$\langle 1^2\rangle_q \to Q_{\langle 2\rangle} + Q_{\langle 1\rangle} + Q_{\langle 0\rangle}$$

and hence

$$Q_{\langle 2\rangle} \circ \langle 1^2\rangle_h = 2Q_{\langle 31\rangle} + 3Q_{\langle 21\rangle} + 2Q_{\langle 4\rangle} + 6Q_{\langle 3\rangle} + 8Q_{\langle 2\rangle} + 4Q_{\langle 1\rangle} + 2Q_{\langle 0\rangle}$$

and thus for *n odd* we have

$$\langle 2\rangle'^{\dagger}\langle 1^2\rangle = 2(2\langle 31\rangle'^{\dagger} + 3\langle 21\rangle'^{\dagger}) + (2\langle 4\rangle'^{\dagger} + 6\langle 3\rangle'^{\dagger} + 8\langle 2\rangle'^{\dagger} + 4\langle 1\rangle'^{\dagger}) + 2\langle 0\rangle'^{\dagger}$$

and for *n even*

$$\langle 2\rangle'^{\dagger}\langle 1^2\rangle = 2\langle 31\rangle'^{\dagger} + 3\langle 21\rangle'^{\dagger} + 2\langle 4\rangle'^{\dagger} + 6\langle 3\rangle'^{\dagger} + 8\langle 2\rangle'^{\dagger} + 4\langle 1\rangle'^{\dagger} + \langle 0\rangle'^{\dagger}.$$

The above algorithm successfully resolves any product $\langle\nu\rangle'^{\dagger}\langle\mu\rangle$. It remains to consider the case where $\langle\nu\rangle^{\dagger} \equiv \langle\nu\rangle' + \langle\tilde{\nu}\rangle'$, which arises when $n-k$ is odd. A study of the difference characters and Morris's theorem 5 (Morris 1962a) shows that the terms in $\langle\nu\rangle'\langle\mu\rangle$ and $\langle\tilde{\nu}\rangle'\langle\mu\rangle$ may be found using the following algorithm.

Algorithm 3.
(1) Evaluate $\langle\nu\rangle'^{\dagger}\langle\mu\rangle$ using algorithm 2.
(2) Divide the coefficients associated with every term found in (1) by two. The integral part of the resulting coefficients is the number of times its corresponding irrep occurs in $\langle\nu\rangle'\langle\mu\rangle$ and in $\langle\tilde{\nu}\rangle'\langle\mu\rangle$. If there is no residue then the resolution is complete.
(3) The only possible residue will be a term $\langle\nu\rangle'^{\dagger} = \langle\nu\rangle' + \langle\tilde{\nu}\rangle$. If the characteristic $\chi^{\langle\mu\rangle}_{\langle\nu\rangle'} = +1$, $\langle\nu\rangle'$ is assigned to $\langle\nu\rangle'\langle\mu\rangle$ and $\langle\tilde{\nu}\rangle'$ to $\langle\nu\rangle'\langle\mu\rangle$, while if $\chi^{\langle\mu\rangle}_{\langle\nu\rangle'} = -1$ the opposite assignment is made.

The characteristics $\chi^{\langle\mu\rangle}_{\langle\nu\rangle'}$ may be readily calculated by first noting that the class $(n-m', (\nu))$ can only involve distinct cycles and

$$\chi^{\langle\mu\rangle}_{\langle\nu\rangle'} = \chi^{[n-m,(\mu)]}_{(n-m',(\nu))}$$

where m and m' are the weights of the partitions (μ) and (ν) respectively. The value of the characteristic $\chi^{[\pi]}_{(\rho)}$ may be found from a theorem due to Littlewood (1950, p 70) (cf Rutherford 1948, p 76).

Theorem. The characteristic $\chi^{[\pi]}_{(\rho)}$ is given by

$$\chi^{[\pi]}_{(\rho)} = \sum_i d_i$$

where there is one term d_i for each way in which the shape π can be built up by making firstly a regular application of ρ_1 spaces, secondly a regular application of ρ_2 spaces, . . . , and lastly a regular application of ρ_h spaces, and where $d_i = (-1)^{t_i}$, t_i being the sum of the numbers of vertical steps in the h applications.

As an example of the application of algorithm 3, we consider the evaluation of $\langle 2\rangle'\langle 1\rangle$ and $\langle\tilde{2}\rangle'\langle 1\rangle$. From use of algorithm 2 we deduce that for *n odd*

$$\langle 2\rangle'^{\dagger}\langle 1\rangle = 2\langle 21\rangle'^{\dagger} + 2\langle 3\rangle'^{\dagger} + 3\langle 2\rangle'^{\dagger} + 2\langle 1\rangle'^{\dagger}$$

while for *n even*

$$\langle 2\rangle'^{\dagger}\langle 1\rangle = 2\langle 21\rangle'^{\dagger} + 4\langle 3\rangle'^{\dagger} + 6\langle 2\rangle'^{\dagger} + 4\langle 1\rangle'^{\dagger}.$$

For *n even* the above result is complete since $\langle 2\rangle'^{\dagger}$ is self-associated. For *n odd* we must have

$$\langle 2\rangle'\langle 1\rangle \supset \langle 21\rangle'^{\dagger} + \langle 3\rangle'^{\dagger} + \langle 2\rangle'^{\dagger} + \langle 1\rangle'^{\dagger}$$

and

$$\langle \tilde{2}\rangle'\langle 1\rangle \supset \langle 21\rangle'^{\dagger} + \langle 3\rangle'^{\dagger} + \langle 2\rangle'^{\dagger} + \langle 1\rangle'^{\dagger}.$$

The residue is $\langle 2\rangle'^{\dagger} = \langle 2\rangle' + \langle \tilde{2}\rangle'$. We need to evaluate $\chi^{\langle 1\rangle}_{\langle 2\rangle'}$. Consider $\chi^{[41]}_{\langle 32\rangle}$; use of Littlewood's theorem yields the diagram

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline 1 \\ \cline{1-1} \end{array}$$

with just one vertical step, and hence we deduce that

$$\chi^{\langle 1\rangle}_{\langle 2\rangle'} = -1.$$

This implies that $\langle \tilde{2}\rangle'$ must be assigned to $\langle 2\rangle'\langle 1\rangle$ and $\langle 2\rangle'$ to $\langle \tilde{2}\rangle'\langle 1\rangle$, and hence for *n odd*

$$\langle 2\rangle'\langle 1\rangle = \langle 21\rangle'^{\dagger} + \langle 3\rangle'^{\dagger} + \langle 2\rangle'^{\dagger} + \langle 1\rangle'^{\dagger} + \langle \tilde{2}\rangle'$$

$$\langle \tilde{2}\rangle'\langle 1\rangle = \langle 21\rangle'^{\dagger} + \langle 3\rangle'^{\dagger} + \langle 2\rangle'^{\dagger} + \langle 1\rangle'^{\dagger} + \langle 2\rangle'.$$

11. Kronecker products of spin irreps

It remains now to develop an algorithm for resolving the Kronecker product of a spin irrep $\langle \mu\rangle'$ with another spin irrep $\langle \nu\rangle'$ into a sum of ordinary irreps. To this end we first compute $\langle \mu\rangle'^{\dagger}\langle \nu\rangle'^{\dagger}$. This may be achieved using the following algorithm.

Algorithm 4.

(1) Expand $\langle \mu\rangle'^{\dagger}$ and $\langle \nu\rangle'^{\dagger}$ as products of the basic spin irrep $\langle 0\rangle'^{\dagger}$ with ordinary irreps to yield

$$\langle \mu\rangle'^{\dagger} \rightarrow \langle 0\rangle'^{\dagger}(g_{\mu}{}^{\pi}\langle \pi\rangle)$$

$$\langle \nu\rangle'^{\dagger} \rightarrow \langle 0\rangle'^{\dagger}(g_{\nu}{}^{\sigma}\langle \sigma\rangle).$$

(2) The product $\langle 0\rangle'^{\dagger} \times \langle 0\rangle'^{\dagger}$ is evaluated for $S_{2\nu+1}$ as

$$\langle 0\rangle'^{\dagger}\langle 0\rangle'^{\dagger} = \sum_{x=0}^{\nu} \langle 1^{x}\rangle^{\dagger} \tag{68a}$$

and for $S_{2\nu}$ as

$$\langle 0\rangle'^{\dagger}\langle 0\rangle'^{\dagger} = 2\sum_{x=0}^{\nu-1} \langle 1^{x}\rangle^{\dagger}. \tag{68b}$$

(3) The calculation is now reduced to the evaluation of Kronecker products of ordinary irreps and may be effected by use of (4).

The expressions given in (68*a*) and (68*b*) follow directly from resolving the Kronecker products of the basic spin irreps of O_n (Littlewood 1950) and then making use of the $O_n \to S_n$ reductions discussed in § 6.

Use of the above algorithm readily leads to the n-independent results

$$\langle 1\rangle'^{\dagger}\langle 1\rangle'^{\dagger} = \langle 0\rangle'^{\dagger 2}(\langle 2\rangle + \langle 1^2\rangle + 2\langle 0\rangle - \langle 1\rangle)$$

$$\langle 2\rangle'^{\dagger}\langle 1\rangle'^{\dagger} = \langle 0\rangle'^{\dagger 2}(\langle 3\rangle + \langle 21\rangle + 2\langle 1\rangle - \langle 2\rangle - 2\langle 0\rangle)$$

with the right-hand side being divided by four for n even. If the above results are specialised to S_7 we find

$$[61]'^{\dagger}[61]'^{\dagger} = 2[7]^{\dagger} + 4[61]^{\dagger} + 6[52]^{\dagger} + 8[51^2]^{\dagger} + 4[43]^{\dagger} + 10[421]^{\dagger} + 8[41^3]^{\dagger} + 4[32^2]^{\dagger} \qquad (69a)$$

$$[52]'^{\dagger}[61]'^{\dagger} = 4[61]^{\dagger} + 8[52]^{\dagger} + 8[51^2]^{\dagger} + 8[43]^{\dagger} + 20[421]^{\dagger} + 12[41^3]^{\dagger} + 12[32^2]^{\dagger}. \qquad (69b)$$

The $[61]'^{\dagger}$ and $[52]'^{\dagger}$ irreps of S_7 constitute pairs of associated irreps. To resolve the products it is necessary to consider the properties of the difference characters introduced in (11).

Consider the product $\langle\mu\rangle'\langle\nu\rangle'$ where $\mu \not\equiv \nu$ and the irreps are associated irreps. Since $\langle\mu\rangle'\langle\nu\rangle'' = \langle\mu\rangle''\langle\nu\rangle' = 0$, we have from consideration of (12*a*) and (12*b*) that

$$\langle\mu\rangle'\langle\nu\rangle' = \langle\mu\rangle'\langle\tilde{\nu}\rangle' = \tfrac{1}{4}\langle\mu\rangle'^{\dagger}\langle\nu\rangle'^{\dagger} \qquad (70)$$

and hence if $\mu \neq \nu$ we may trivially resolve the Kronecker products. Thus from (69*b*) we have

$$[52]'[61]' = [52]' \times [\widetilde{61}]'$$
$$= [61]^{\dagger} + 2[52]^{\dagger} + 2[51^2]^{\dagger} + 2[43]^{\dagger} + 5[521]^{\dagger} + 3[41^3]^{\dagger} + 3[32^2]^{\dagger}.$$

When $\mu \equiv \nu$ we have

$$\langle\mu\rangle'\langle\mu\rangle' = \tfrac{1}{4}[\langle\mu\rangle'^{\dagger 2} + \langle\mu\rangle'\,''^{2}] \qquad (71)$$

$$\langle\mu\rangle'\langle\tilde{\mu}\rangle' = \tfrac{1}{4}[\langle\mu\rangle'^{\dagger 2} - \langle\mu\rangle'\,''^{2}] \qquad (72)$$

$$= \langle\tilde{0}\rangle(\langle\mu\rangle'\langle\mu\rangle'). \qquad (73)$$

The problem is solved once $\langle\mu\rangle'\,''^{2}$ is resolved. Consider the character of $\langle\mu\rangle'\,''^{2}$. This will have non-zero characteristics only for the classes of $(\lambda_1 \ldots \lambda_k)$ and $(\lambda_1 \ldots \lambda_k)'$. Let $\langle\mu\rangle'\,''^{2} = g_{\mu\mu}{}^{\rho}\langle\rho\rangle$; in each of these classes we have

$$\chi^{\langle\mu\rangle''}_{(\mu)} = 2\mathrm{i}^{n-k+1}\lambda_1\lambda_2\ldots\lambda_k = g_{\mu\mu}{}^{\rho}\chi^{\langle\rho\rangle}_{(\mu)} \qquad (74)$$

where

$$g_{\mu\mu}{}^{\rho} = 2\mathrm{i}^{n-k+1}\chi^{\langle\rho\rangle}_{(\mu)}$$

and $n-k+1$ is even, so

$$g_{\mu\mu}{}^{\rho} = 2\mathrm{i}^{n-k+1}\chi^{[n-m',\rho]}_{(\lambda)}. \qquad (75)$$

The characteristics $\chi^{[n-m',\rho]}_{(\lambda)}$ may be found from Littlewood's theorem.

In the case of S_7 we easily find

$$[61]'\,''^{2} = 2([1^7] - [7] + [52] - [2^21^3] - [421] + [321^2])$$

and hence from (69a) and (71) we deduce that

$$[61]'[61]' = [61]^{\dagger} + [52]^{\dagger} + [52] + 2[51^2]^{\dagger} + 2[41^3]^{\dagger}$$
$$+ [43]^{\dagger} + 2[421]^{\dagger} + [421] + [32^2]^{\dagger} + [\tilde{7}]$$
$$[61]'[\widetilde{61}]' = [61]^{\dagger} + [52]^{\dagger} + [\widetilde{52}] + 2[51^2]^{\dagger} + 2[41^3]^{\dagger}$$
$$+ [43]^{\dagger} + 2[421]^{\dagger} + [\widetilde{421}] + [32^2]^{\dagger} + [7].$$

As a consequence of the preceding, it is evident that any Kronecker product involving spin irreps may be systematically resolved. Explicit determination of the difference characters is only required for the special case of $(\lambda) \equiv (\mu)$.

12. Plethysms for spin irreps

We now consider the problem of resolving the Kronecker square of the spin irreps of S_n into its symmetric and antisymmetric terms. We first note that for $O_{2\nu+1}$ we have

$$\Delta \otimes \{2\} = [1^{2\nu}] + [1^{2\nu-4}] + [1^{1\nu-8}] + \ldots \tag{76a}$$
$$\Delta \otimes \{1^2\} = [1^{2\nu-2}] + [1^{2\nu-6}] + [1^{2\nu-10}] + \ldots \tag{76b}$$

for $\nu = 0, 1 \pmod 4$, while

$$\Delta \otimes \{2\} = [1^{2\nu-2}] + [1^{2\nu-6}] + [1^{2\nu-10}] + \ldots \tag{77a}$$
$$\Delta \otimes \{1^2\} = [1^{2\nu}] + [1^{2\nu-4}] + [1^{2\nu-8}] + \ldots \tag{77b}$$

for $\nu = 2, 3 \pmod 4$.

Consideration of the restriction $O_{2\nu+1} \downarrow S_{2\nu+1}$ then leads to

$$\langle 0\rangle'^{\dagger} \otimes \{2\} = \langle 1^{2\nu}\rangle + \langle 1^{2\nu-1}\rangle + \langle 1^{2\nu-4}\rangle + \langle 1^{2\nu-5}\rangle + \ldots \tag{78a}$$
$$\langle 0\rangle'^{\dagger} \otimes \{1^2\} = \langle 1^{2\nu-2}\rangle + \langle 1^{2\nu-3}\rangle + \langle 1^{2\nu-6}\rangle + \langle 1^{2\nu-7}\rangle + \ldots \tag{78b}$$

for $\nu = 0, 1 \pmod 4$ and

$$\langle 0\rangle'^{\dagger} \otimes \{2\} = \langle 1^{2\nu-2}\rangle + \langle 1^{2\nu-3}\rangle + \langle 1^{2\nu-6}\rangle + \langle 1^{2\nu-7}\rangle + \ldots \tag{79a}$$
$$\langle 0\rangle'^{\dagger} \otimes \{1^2\} = \langle 1^{2\nu}\rangle + \langle 1^{2\nu-1}\rangle + \langle 1^{2\nu-4}\rangle + \langle 1^{2\nu-5}\rangle + \ldots \tag{79b}$$

for $\nu = 2, 3 \pmod 4$.

In an exactly similar way we find for $S_{2\nu}$

$$\langle 0\rangle'^{\dagger} \otimes \{2\} = \langle 1^{2\nu-1}\rangle + 2\langle 1^{2\nu-2}\rangle + \langle 1^{2\nu-3}\rangle + \langle 1^{2\nu-5}\rangle + 2\langle 1^{2\nu-6}\rangle + \langle 1^{2\nu-7}\rangle + \ldots \tag{80a}$$
$$\langle 0\rangle' \otimes \{2\} = \langle 1^{2\nu-1}\rangle + \langle 1^{2\nu-3}\rangle + 2\langle 1^{2\nu-4}\rangle + \langle 1^{2\nu-5}\rangle + \langle 1^{2\nu-7}\rangle + 2\langle 1^{2\nu-8}\rangle + \langle 1^{2\nu-9}\rangle + \ldots \tag{80b}$$

for $\nu = 1, 2 \pmod 4$ and

$$\langle 0\rangle'^{\dagger} \otimes \{2\} = \langle 1^{2\nu-1}\rangle + \langle 1^{2\nu-3}\rangle + 2\langle 1^{2\nu-4}\rangle + \langle 1^{2\nu-5}\rangle + \langle 1^{2\nu-7}\rangle + 2\langle 1^{2\nu-8}\rangle + \langle 1^{2\nu-8}\rangle + \ldots \tag{81a}$$
$$\langle 0\rangle'^{\dagger} \otimes \{1^2\} = \langle 1^{2\nu-1}\rangle + 2\langle 1^{2\nu-2}\rangle + \langle 1^{2\nu-3}\rangle + \langle 1^{2\nu-5}\rangle + 2\langle 1^{2\nu-6}\rangle + \langle 1^{2\nu-7}\rangle + \ldots \tag{81b}$$

for $\nu = 0,3 \pmod 4$.

A representation λ is said to be *orthogonal* if the identity irrep occurs in the symmetric part of the Kronecker square (i.e. in $\lambda\otimes\{2\}$) or *symplectic* if it occurs in the antisymmetric part (i.e. in $\lambda\otimes\{1^2\}$). If the identity irrep fails to occur in either $\lambda\otimes\{2\}$ or $\lambda\times\{1^2\}$ then λ corresponds to a complex irrep. This inspection of (78*a*)–(79*b*) shows that for $S_{2\nu+1}$ the basic spin irrep $\langle 0\rangle'^{\dagger}$ has the property

if $\quad \nu=0, 3 (\bmod 4) \quad \langle 0\rangle'^{\dagger}$ is orthogonal

while

if $\quad \nu=1, 2 (\bmod 4) \quad \langle 0\rangle'^{\dagger}$ is symplectic.

For $S_{2\nu+1}$ the basic spin irrep is always self-associated and never complex.

In the case of $S_{2\nu}$ the basic spin irrep is an associated pair of irreps $\langle 0\rangle'^{\dagger}=\langle 0\rangle'+\langle\tilde{0}\rangle'$, and it is necessary to use difference characters to complete the analysis of the Kronecker square. We have

$$\langle 0\rangle'\cdot\langle 0\rangle'=\tfrac{1}{4}[\langle 0\rangle'^{\dagger 2}+\langle 0\rangle''^{2}] \tag{82}$$

where

$$\langle 0\rangle'^{\dagger 2}=2\sum_{x=0}^{\nu-1}\langle 1^x\rangle^{\dagger} \tag{83}$$

$$\langle 0\rangle'\,''^{2}=2\mathrm{i}^n\sum_{x=0}^{2\nu-1}(-1)^x\langle 1^x\rangle, \tag{84}$$

a result that comes from (74) and noting that $\chi_{(n)}^{[n-m',p]}=(-1)^x$ if $p=1^x$ or else is zero.

The square of the difference character $\langle 0\rangle'\,''^{2}$ may be analysed into its symmetric and antisymmetric parts to give

$$\begin{aligned}\langle 0\rangle'\,''\otimes\{2\}&=(\langle 0\rangle'-\langle\tilde{0}\rangle')\otimes\{2\}\\&=\langle 0\rangle'^2-\langle 0\rangle'\langle\tilde{0}\rangle'\end{aligned} \tag{85a}$$

and

$$\langle 0\rangle'\,''\otimes\{1^2\}=\langle 0\rangle'^2-\langle 0\rangle'\langle\tilde{0}\rangle' \tag{85b}$$

and hence

$$\langle 0\rangle'\,''\otimes\{2\}=\langle 0\rangle'\,''\otimes\{1^2\}=\langle 0\rangle'\,''^{2}/2. \tag{86}$$

The Kronecker square of the basis spin irreps $\langle 0\rangle'$ and $\langle\tilde{0}\rangle'$ of $S_{2\nu}$ may be evaluated using (82), (83) and (84). The resolution of the Kronecker square into its symmetric and antisymmetric terms then follows by noting that

$$\langle 0\rangle'\otimes\{2\}=\tfrac{1}{2}[\langle 0\rangle'^{\dagger}\otimes\{2\}+\langle 0\rangle''\otimes\{2\}-\langle 0\rangle'^{2}] \tag{87a}$$

$$\langle 0\rangle'\otimes\{1^2\}=\tfrac{1}{2}[\langle 0\rangle'^{\dagger}\otimes\{1^2\}+\langle 0\rangle''\otimes\{1^2\}-\langle 0\rangle'^{2}]. \tag{87b}$$

Noting (85*a*)–(86) leads to the conclusion that

if $\quad \nu=1, 3 (\bmod 4) \quad \langle 0\rangle'$ and $\langle\tilde{0}\rangle'$ are complex $\quad$ (88*a*)

while

if $\quad \nu=0 (\bmod 4) \quad \langle 0\rangle'$ and $\langle\tilde{0}\rangle'$ are orthogonal $\quad$ (88*b*)

or

$\nu=2 (\bmod 4) \quad \langle 0\rangle'$ and $\langle\tilde{0}\rangle'$ are symplectic. $\quad$ (88*c*)

Once the plethysms of the basic spin irreps of S_n are known, we may evaluate plethysms for any spin irrep of S_n, since we can always reduce any spin irrep to the product of the basic spin with a sum of ordinary irreps. With the Kronecker square of the basic spin irreps resolved as above, we can now in principle resolve the Kronecker square of any irrep of S_n. The resolution of higher Kronecker powers would first require the resolution of higher powers of the basic spin irreps. In these cases the $O_n \downarrow S_n$ embedding can be exploited to yield the resolution of the appropriate powers of the basic spin irreps.

13. Classification of spin irreps

The spin irreps of S_n may be classified as to their symplectic, orthogonal or complex characters by use of the classification found for the basic spin irreps and remembering that the ordinary irreps of S_n are all orthogonal. The following algorithm leads to a complete classification.

Algorithm 5.

(1) If $(n-k+1)/2$ is odd then the spin irrep $[\lambda_1 \dots . \lambda_k]'$ is complex.

(2) If $(n-k+1)/2$ or $(n-k)$ are even for $n=2\nu+1$ we have for the spin irreps

$\nu = 0, 3 (\text{mod } 4)$	orthogonal
$\nu = 1, 2 (\text{mod } 4)$	symplectic

while for $n = 2\nu$ we have

$\nu = 0, 1 (\text{mod } 4)$	orthogonal
$\nu = 2, 3 (\text{mod } 4)$	symplectic.

14. Concluding remarks

The main thrust of this paper has been to develop a reduced n-independent notation for the spin irreps of the symmetric group and then to establish a series of simple algorithms to enable their properties to be evaluated in a largely n-independent manner. The need for explicit character tables has effectively been eliminated. Much of this work has a direct bearing on the problem of constructing $3jm$ and nj symbols for the symmetric group, particularly since every irrep of a given S_n must arise in some power of the basic spin irrep of S_n. Starting with the basic spin irrep, it should be possible systematically to build up the $3jm$ symbols involving both spin and ordinary irreps of S_n.

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